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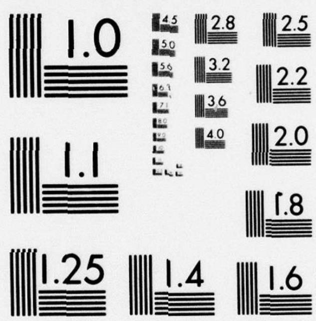

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A Method for Treating the Sheath Size in the Langmuir Mott-Smith Equations

CHRISTOPHER SHERMAN

2 June 1978

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FOR THE COMMANDER

John Howard

Chief Scientist

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An argument is presented which justifies the simultaneous use of both the Langmuir Mott-Smith orbit limited, and the diode space charge limited equations for current collection by electrostatic probes. From these two equations the sheath size can be eliminated and the current to a probe obtained for a large range of values of probe potential and Debye length. Numerical solutions are presented for both the spherical and cylindrical cases. Asymptotic properties are considered.

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Preface

The author wishes to thank Pat Bench for programming the numerical solutions.

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A Method for Treating the Sheath Size in the Langmuir Mott-Smith Equations

1. INTRODUCTION

Recent work in probe theory[1] has been largely oriented towards collection of ions from laboratory plasmas in which the electron temperature greatly exceeds ion temperatures. In the ionosphere, electron temperatures are at most only ~ 2 to 3 times ion temperatures and are often even lower than this. In these circumstances the original Langmuir Mott-Smith (LMS) equations[2] are applicable not only to electrons but also possibly to ions, and if some means of treating a sheath of unknown size were available, the more recent (and more complex) theories with the attendant difficulties in use could be dispensed with. This is the problem which is addressed in the present paper. The treatment developed here will also be applicable to the charging of vehicles by emission of beams of charged particles into the ionosphere, since with the interchange of roles of dependent and independent variables, the current-voltage relationships in this case may be identical with those governing the behavior of electric probes.

(Received for publication 2 June 1978)

1. Swift, J.D. and Schwar, M.J.R. (1969) Electrical Probes for Plasma Diagnostics, American Elsevier, pp 69-72.
2. Suits, C.G., Ed. (1961) The Collected Works of Irving Langmuir, Vol. 4, Pergamon Press.

Although the LMS formulas have been discussed extensively in the literature, there still seems to be some question concerning both the limits of their applicability and also their relationship to the Langmuir space charge limited (SCL) equations[3] for diodes. We are not referring to limitations such as the shape of potential curves or relative values of positive and negative particle temperatures, but rather to more obvious scaling effects. In much previous work, the limiting cases of the LMS equations are discussed solely in terms of the variation in sheath size. Here, the limiting cases are discussed in terms of combinations of sheath size and potential, as they actually appear in the LMS equations. As a result of this treatment, it will appear that the relationship between the LMS and SCL equations is not necessarily one of mutual exclusion but rather, often, of complementarity as far as domains of applicability are concerned. From the two equations, the sheath size can then be eliminated and an expression obtained for the current collected in terms of voltage and Debye length which is approximately valid for a large range of values of these two parameters. The arguments leading to these results are given in the following sections.

2. THE SPHERICAL CASE

We first write down the two equations for a spherical collector in a conveniently nondimensionalized form thus:[4] (See also previous references.)

$$J = \rho^2 \left[1 - (1 - 1/\rho^2)\, e^{-\phi/(\rho^2-1)}\right] \tag{1}$$

$$J = C^2 \phi^{3/2}/[\alpha(\rho)]^2 \left(1 + \frac{2.66}{\sqrt{\phi}}\right) \quad . \tag{2}$$

The definitions of the variables appearing in Eqs. (1) and (2) are:

$$J = I \left[4\pi r_p^2\, e\, N_\infty \left(\frac{kT}{2\pi m}\right)^{1/2}\right]^{-1}$$

$$\rho = r_s/r_p$$

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3. Suits, C.G., Ed. (1961) The Collected Works of Irving Langmuir, Vol. 3, Pergamon Press.

4. Chen, Francis F. (1965) Plasma Diagnostic Techniques, Huddlestone and Leonard, Eds, Academic Press.

$$\phi = \frac{e V_p}{kT}$$

$$C = \frac{2\sqrt{2\sqrt{\pi}}}{3} \lambda_D / r_p$$

and

r_p = probe radius

r_s = sheath radius

λ_D = Debye length

V_p = probe potential

e = electronic charge

N_∞ = ambient undisturbed charged particle density

T = (Maxwellian) temperature of ambient charged particles

m = mass of collected particle

k = Boltzmann constant

I = Probe current

We note that α is a known function of ρ, so that Eqs. (1) and (2) are two equations in the four variables C, ϕ, ρ and J. Concerning Eq. (1), the LMS equation, there is in its derivation no assumption, aside from an indirect one related to the shape of the potential, relative to the size of the sheath. There is only an assumption that a sheath edge exists, beyond which the plasma properties assume their undisturbed values. Thus, in the absence of other restrictions, Eq. (1) is expected to be valid for any sheath size. Concerning Eq. (2), the SCL equation, the limiting assumption is that all particles which leave the sheath edge arrives at the probe. Now, when $\phi/(\rho^2 - 1) \gg 1$ in Eq. (1), the latter reduces to $J = \rho^2$: that is, all the particles leaving the sheath edge reaches the probe. In fact, when the more severe condition $\phi/\rho^2 \gg 1$ holds, the motion of charged particles to the probe is approximately radial. The key to the simultaneous validity of Eqs. (1) and (2) is thus the criterion $\phi/(\rho^2 - 1) \gg 1$. When $\phi/(\rho^2 - 1) \ll 1$ on the other hand, Eq. (1) reduces to $J = 1 + \phi$ which is independent of ρ. We thus arrive at a situation in which solving the two equations simultaneously for J as a function of ϕ and C yields results which are valid both for $\phi/(\rho^2 - 1) \gg 1$ and for $\phi/(\rho^2 - 1) \ll 1$. Since $\phi/(\rho^2 - 1) = 1$ is not a singular point for either equation and further since Eq. (2) is not too bad an approximation at $\phi/(\rho^2 - 1) = 1$, it is not unreasonable to assume that values of J obtained for $\phi/(\rho^2 - 1) \cong 1$ are also approximately correct. As far

as values of ρ are concerned, however, this is clearly not the case; the values of ρ obtained are only accurate for $\phi/(\rho^2 - 1) \gg 1$. We recognize the heuristic nature of these arguments and explore the consequences of their assumed validity.

To solve Eqs. (1) and (2) we form:

$$F(\rho) = \rho^2 \left[1 - (1 - 1/\rho^2)\, e^{-\phi/(\rho^2-1)}\right] [\alpha(\rho)]^2 - C^2 \phi^{3/2} \left(1 + \frac{2.66}{\sqrt{\phi}}\right) = 0 \quad . \quad (3)$$

Using the Newton-Raphson method Eq. (3) is solved for ρ, and J is then obtained from either Eq. (1) or (2). Details of this solution are given in the appendix.

The results of these calculations are shown in Figures 1 and 2. In Figure 2, values of $\rho - 1$ above and to the left of the curve labeled $\rho = \sqrt{1 + \phi}$ are available, but as noted the calculation does not yield valid results for ρ in this region. As the curve $\rho = \sqrt{1 + \phi}$ is approached, the accuracy of the values for ρ is reduced, while values away from this curve will have increasingly better accuracy. The dashed line in Figure 1 is for $J = 1 + \phi(1 - 1/e)$ and is hence the equivalent of the curve $\rho = \sqrt{1 + \phi}$ in Figure 2.

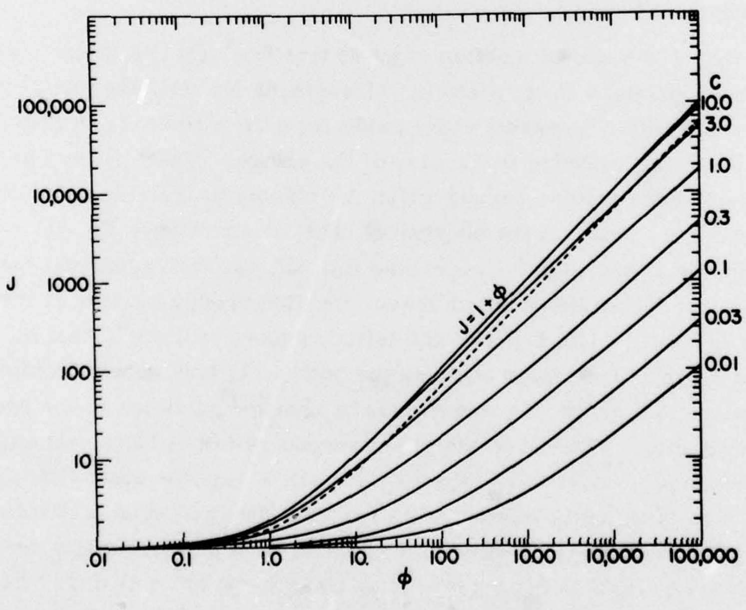

Figure 1. Nondimensionalized Current J vs Nondimensionalized Potential ϕ for Various Values of the Parameter C for a Spherical Collector

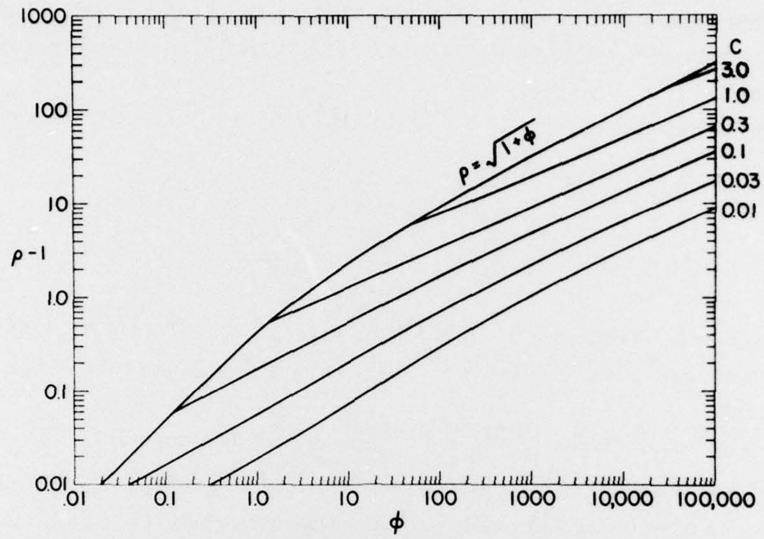

Figure 2. Nondimensionalized Sheath Size ρ vs Nondimensionalized Potential ϕ for Various Values of the Parameter C for a Spherical Collector

3. ASYMPTOTIC PROPERTIES

We investigate the dependence of J and ρ on ϕ and C as $\phi \to \infty$, and as $\phi \to 0$. Note the difference between this asymptotic limit and the limits commonly discussed in probe theory.[1] In the latter, ϕ is held fixed while $\rho \to \infty$ or 0; here, $\phi \to \infty$ or 0 while ρ varies as an implicit function of ϕ. The procedure followed here is the more useful, since the variables under direct experimental control are ϕ and C, not ρ.

As $\phi \to \infty$, we may assume that $\rho \to \infty$ so that $\alpha^2 \to (1.11)^{3/2} \rho^{3/2}$ (See Appendix, Eq. (12)). Thus, Eq. (2) may be replaced in this limit by

$$J = \frac{C^2 \phi^{3/2}}{(1.11)^{3/2} \rho^{3/2}} \quad . \tag{4}$$

There are now three possibilities.

(1) As ϕ, $\rho \to \infty$, $\phi/(\rho^2 - 1) \to \infty$

(2) As ϕ, $\rho \to \infty$, $\phi/(\rho^2 - 1) \to 0$

(3) As ϕ, $\rho \to \infty$, $\phi/(\rho^2 - 1) \to \underline{\overline{k}} \neq 0, \infty$.

We examine each of these in turn to see if they can represent self-consistent solutions of Eqs. (1) and (4).

(1) In this case $\phi \gg \rho^2 - 1$, so that $J = \rho^2$ (Eq. (1)), $\rho^{7/2} = C^2 \phi^{3/2}/(1.11)^{3/2}$ (Eq. (4)), $\phi = 1.11\, \rho^{7/3}/C^{4/3}$, $\phi/(\rho^2 - 1) \cong 1.11\, \rho^{1/3}/C^{4/3} \to \infty$ and this is self-consistent.

We have for this case

$$\rho = C^{4/7} \phi^{3/7}/(1.11)^{3/7} \tag{5}$$

$$J = C^{8/7} \phi^{6/7}/(1.11)^{6/7} \quad . \tag{6}$$

(2) $\rho^2 - 1 \gg \phi$. Then, $J = 1 + \phi \cong \phi$ (Eq. (1)), $\rho^{3/2} = C^2 \phi^{1/2}/(1.11)^{3/2}$ (Eq. (4)), $\phi = (1.11)^3 \rho^3/C^4$, $\phi/(\rho^2 - 1) \cong (1.11)^3 \rho/C^4 \to \infty$ and this is not self consistent.

(3) $\phi = \underline{\bar{k}}(\rho^2 - 1)$. Then, $\lim_{\phi \to \infty} e^{-\phi/\rho^2 - 1} = e^{-\underline{\bar{k}}} \equiv B$ a constant, $J = \rho^2(1 - B) + B$ (Eq. (1)), $\rho^2(1 - B) + B = C^2/(1.11)^{3/2} \phi^{3/2}/\rho^{3/2}$, and for ρ large this is, outside of the factor $(1 - B)$ identical with case (1), which leads to $\phi/(\rho^2 - 1) \to \infty$ and is hence not self-consistent.

Thus, only case (1) is self-consistent and Eqs. (5) and (6) must be the asymptotic limits for $\phi \to \infty$.

The two conditions necessary for Eqs. (5) and (6) to be valid, namely $1.11\rho \gg 1.64$ and $\phi/(\rho^2 - 1) \gg 1$ are asymptotically equivalent to $(1.11)^{4/7} C^{4/7} \phi^{3/7} \gg 1.64$ and $(1.11)^{6/7} \phi^{1/7}/C^{8/7} \gg 1$ respectively. The accuracy of the asymptotic approximation thus increases with increasing ϕ, but may either increase or decrease with increasing C. This behavior is confirmed by a comparison of numerical and asymptotic values. The agreement is best at $C \cong 1$ in which case the asymptotic value is accurate to within 2 percent at $\phi = 10^4$ and to within 0.4 percent at $\phi = 10^6$ for both J and ρ.

The opposite limit, $\phi \to 0$ can also be investigated using Eq. (2) since the latter is derived for charged particles leaving the emitter surface with nonzero initial velocity, and is hence valid for small values of ϕ. Although this case is not of great interest practically, a consistency analysis similar to that performed for the $\phi \to \infty$ limit is easily carried out and shows that the only consistent results for $\phi \to 0$ are

$$\phi/(\rho^2 - 1) \to \frac{\phi^{1/2}}{2\sqrt{2.66}\, C} \to 0$$

and hence that $\lim_{\phi \to 0} J = 1 + \phi$.

For fixed ϕ, the approximation becomes better with increasing values of C. Thus, for example, for $\phi = 0.01$, the asymptotic value of J is 1.01 and the numerical values are as follows:

C	0.01	0.03	0.1	0.3	1.0	3	10	30	100
J	1.003	1.006	1.009	1.010	1.010	1.010	1.010	1.010	1.010

In this limit ($\phi \to 0$) since $\phi/(\rho^2 - 1) \ll 1$, there is no valid approximation for ρ.

4. THE CYLINDRICAL CASE

The argument for the cylindrical case is not as clearcut as is that for the sphere. The LMS equation for the cylinder contains ϕ in three combinations with ρ and an addition assumption is needed to complete the argument, namely $\phi \geqq 1$. Nevertheless, values of ϕ for which this holds are the ones of greatest practical interest.

The two applicable equations for the cylindrical case are (previous references):

$$J = \rho \operatorname{erf} \sqrt{\frac{\phi}{\rho^2 - 1}} + e^{\phi} \left[1 - \operatorname{erf} \sqrt{\frac{\rho^2 \phi}{\rho^2 - 1}} \right] \tag{7}$$

$$J = C^2 \frac{\phi^{3/2}}{[\beta(\rho)]^2} \left(1 + \frac{2.66}{\sqrt{\phi}} \right) \tag{8}$$

where

$$J = I \left[2\pi L r_p e N_\infty \sqrt{\frac{kT}{2\pi m}} \right]^{-1} .$$

L = probe length and all other quantities are as previously defined for the sphere.

Now, let $\phi \geqq 1$, and to start with, $\phi/(\rho^2 - 1) \ll 1$. Then, $\rho^2\phi/(\rho^2 - 1) \cong \phi$ and Eq. (7) becomes

$$J \cong \rho \frac{2}{\sqrt{\pi}} \sqrt{\frac{\phi}{\rho^2 - 1}} + e^{\phi} (1 - \operatorname{erf} \sqrt{\phi})$$

$$\cong \frac{2}{\sqrt{\pi}} \sqrt{\phi} + e^{\phi} (1 - \operatorname{erf} \sqrt{\phi})$$

which is independent of ρ. In the opposite case, namely $\phi/(\rho^2 - 1) \gg 1$, we must also have $\rho^2\phi/(\rho^2 - 1) \gg 1$ and hence $J \cong \rho$. Thus, with the restriction $\phi \geqq 1$, the same reasoning as applied to the spherical case applies here, and the two Eqs. (7) and (8) may be solved simultaneously to obtain J and ρ as functions of ϕ and C. Again, the solutions for ρ are only valid when $\phi/(\rho^2 - 1) \gg 1$.

To solve Eqs. (7) and (8) we form

$$F = \rho \operatorname{erf}\sqrt{\frac{\phi}{\rho^2 - 1}} + e^{\phi}\left(1 - \operatorname{erf}\sqrt{\frac{\rho^2\phi}{\rho^2 - 1}}\right) - \frac{C^2\,\phi^{3/2}}{[\beta(\rho)]^2}\left(1 + \frac{2.66}{\sqrt{\phi}}\right) = 0 \qquad (9)$$

solve this for ρ, and then obtain J from either Eq. (7) or (8). The details of the numerical solution are again given in the appendix.

The results of these calculations are shown in Figures 3 and 4. It is seen that for the cylindrical collector, the solutions for ρ have a domain of validity which differs markedly from the domain of validity of the sphere.

The asymptotic properties of the solution for the cylinder may be obtained by considerations of self-consistency in the same manner as was done for the sphere. The results of such considerations are again that there is only one self-consistent solution for the two equations and for $\phi \to \infty$, this is that $\phi/(\rho^2 - 1) \to 0$, and

$$\lim_{\phi \to \infty} J = \frac{2}{\sqrt{\pi}}\sqrt{\phi} + \frac{1}{\sqrt{\pi\phi}} = \frac{2}{\sqrt{\pi}}\sqrt{\phi} \quad . \qquad (10)$$

The asymptotic expression for ρ may also be formally written down but is meaningless since solutions for ρ are only valid when $\phi/(\rho^2 - 1) \gg 1$, while the asymptotic forms are only valid for $\phi/(\rho^2 - 1) \ll 1$. The asymptotic expression for ρ is thus valid nowhere.

Note the radical difference in behavior between the spherical and cylindrical cases: for the sphere,

$$\lim_{\phi \to \infty} \phi/(\rho^2 - 1) \to \infty$$

but for the cylinder,

$$\lim_{\phi \to \infty} \phi/(\rho^2 - 1) \to 0 \quad .$$

The high voltage limit of the current for the sphere is thus space charge limited, while for the cylinder the high voltage limit is orbit limited.

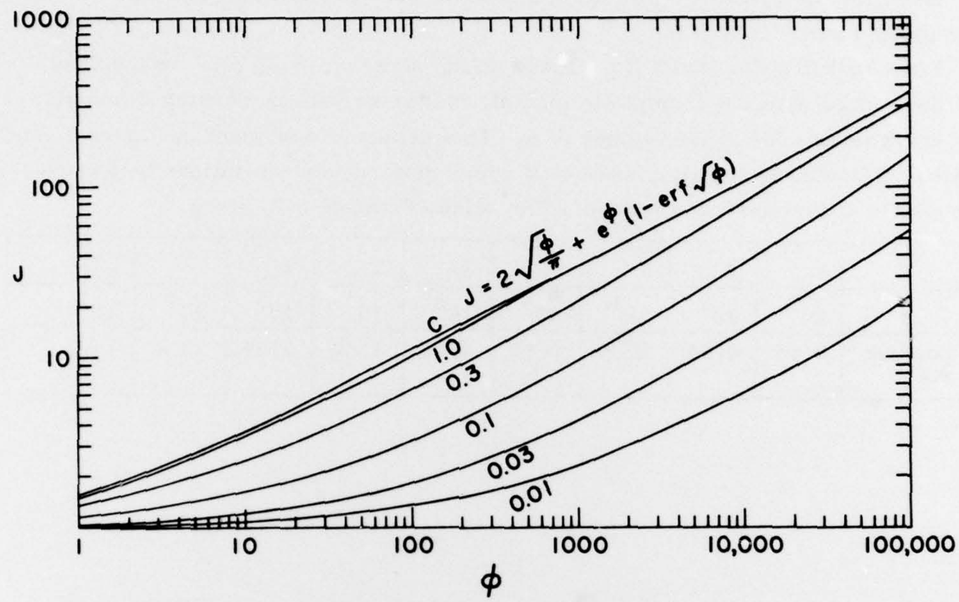

Figure 3. Nondimensionalized Current J vs Nondimensionalized Potential ϕ for Various Values of Parameter C for a Cylindrical Collector

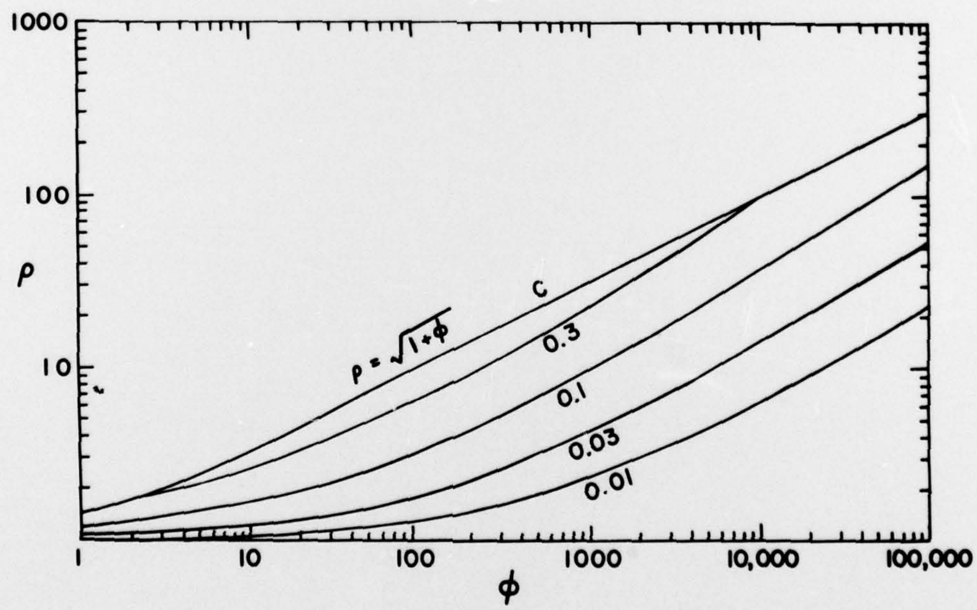

Figure 4. Nondimensionalized Sheath Size ρ vs Nondimensionalized Potential ϕ for Various Values of Parameter C for a Cylindrical Collector

Since for the cylinder we must in any event have $\phi \geqq 1$, there can be no asymptotic forms for $\phi \rightarrow 0$.

The conditions for which Eq. (10) is valid, namely $\phi \gg 1$, $\phi/\rho^2 \ll 1$ indicate that the accuracy of the asymptotic formula increases with increasing C and also with increasing ϕ for higher values of ϕ. This behavior is evident in Figure 3, and is also confirmed by the numbers below which give percent deviations between asymptotic and numerical values of J for selected values of C and ϕ.

C	3.	3.	3.	3.	1.	1.	1.	.3	.3
ϕ	10^2	10^3	10^4	10^5	10^3	10^4	10^5	10^4	10^5
percent dev J	0.89	0.14	0.00	0.00	4.23	1.34	0.34	36.4	14.2

Appendix A

Formulas for Numerical Calculations

Listed below for reference purposes are the expressions for α, α', β, β' and F' used in the Newton-Raphson solutions of Eqs. (3) and (9). (Primes denote derivatives with respect to ρ, and $\gamma = \ln \rho$.)

(1) Spherical Case. The values of $\alpha(\rho)$ needed to solve Eq. (3) are tabulated, but we used a series[1]

$$\alpha_1 = \gamma + 0.3\,\gamma^2 + 0.75\,\gamma^3 + 0.143182\,\gamma^4 + 0.0021609\,\gamma^5 + 0.00026791\,\gamma^6 \tag{A1}$$

for values of $\rho \leqq 10.0$ and a formula[2]

$$\alpha_2 = (1.11\rho - 1.64)^{3/4} \tag{A2}$$

For values of $\rho > 10.0$. At $\rho = 10$, Eq. (A1) yields $\alpha_1^2 = 29.06$, Eq. (A2) yields $\alpha_2^2 = 29.09$, and the tabulated value is $\alpha^2 = 29.19$. The value of ρ is tested at each iteration in the Newton-Raphson process and α_1 or α_2 chosen depending on whether $\rho \leq 10$ or $\rho > 10$. The other expressions needed for the spherical case are:

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1. Suits, C. G., Ed. (1961) The Collected Works of Irving Langmuir, Vol. 4, Pergamon Press.
2. Suits, C. G., Ed. (1961) The Collected Works of Irving Langmuir, Vol. 3, Pergamon Press.

$$F' = \rho^2 \left[1 - (1 - 1/\rho^2)\, e^{-\phi/(\rho^2-1)}\right] \; 2\,\alpha\,\alpha'$$

$$+ 2\,\alpha^2\,\rho \left[1 - (\phi/(\rho^2 - 1) + 1)\, e^{-\phi/(\rho^2-1)}\right]$$

$$\alpha_1' = \frac{1}{\rho}\,[1.0 + 0.6\,\gamma + 0.225\,\gamma^2 + 0.0572728\,\gamma^3 + 0.0108045\,\gamma^4 + 0.00160746\,\gamma^5]$$

$$\alpha_2' = 0.8725\,(1.11\rho - 1.64)^{-1/4} \quad .$$

(2) Cylindrical Case. For $\beta(\rho)$ we used[2]

$$\beta_1 = \gamma + 0.40\,\gamma^2 + 0.0916667\,\gamma^3 + 0.0142424\,\gamma^4 + 0.0016793\,\gamma^5 + 0.0001612\,\gamma^6 \tag{A3}$$

for $\rho \leq 16$ and

$$\beta_2^2 = 4.6712\rho(\log_{10}\rho - 0.1505)^{3/2} \tag{A4}$$

for $\rho > 16$. When $\rho = 16$, Eq. (A3) gives $\beta_1^2 = 80.83$, Eq. (A4) gives $\beta_2^2 = 80.83$, and the tabulated value is 81.203. The remaining expressions for the cylindrical case are:

$$F' = -2/\sqrt{\pi}\; e^{-\phi/(\rho^2-1)}\; \sqrt{\phi}/\sqrt{\rho^2 - 1} + \mathrm{erf}\sqrt{\frac{\phi}{\rho^2 - 1}}$$

$$+ C^2\,\phi^{3/2}\left(1 + \frac{2.66}{\sqrt{\phi}}\right)\frac{2}{\beta^3}\;\beta'$$

$$\beta_1' = \frac{1}{\rho}\,[1 + 0.8\,\gamma + 0.275\,\gamma^2 + 0.0569697\,\gamma^3 + 0.0083965\,\gamma^4 + 0.0009672\,\gamma^5]$$

$$\beta_2' = 2.3356\,\frac{1}{\beta_2}\,[0.65144\,(\log_{10}\rho - 0.1505)^{1/2} + (\log_{10}\rho - 0.1505)^{3/2}]$$